Three-dimensional nonlinear states in the Blasius boundary layer

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Keywords: Hydrodynamic stability, Transition to turbulence.

SUMMARY. Nonlinear three-dimensional coherent structures in a parallel boundary layer flow are presented. The solutions are discovered using the self-sustaining process (F. Waleffe, *Phys. Fluids* **9**, pp. 883-900, 1997). Using this technique a body force is added to the Navier-Stokes equations; the relevant solutions are discovered through performing a continuation in the amplitude of the body force to render it of vanishing amplitude. Some solutions found have a spanwise spacing between the low speed streaks of the same value as has been observed in experimental turbulence.

1 INTRODUCTION

Turbulence and the transitional phase is usually a problem for engineering applications due to high skin friction and large pressure fluctuations, thus decreasing the fluid transport and may also cause damage due to vibrations. Understanding this phenomena would prove much helpful for any practical or everyday occurrence of fluid dynamics and may lead to ideas on how to control it. Even though the past twenty years have witnessed progress on understanding transition to turbulence, the reason for the long time-span in making advancement on this field of research, from the original experiments by Osborne Reynolds in 1883 [1], to the incomplete understanding of today, is related to the nonlinearity of the momentum equations. In the end of the 19th century Henri Poincaré [2] had established that chaotic dynamics is generated by the interplay of locally unstable states and the interweaving of their global stable and unstable manifolds. Some first important step on how a fluid flow becomes turbulent were laid by the theory of Landau [3]. Here to describe the route of transition to turbulence the laminar flow is thought to go through an infinite sequences of bifurcations or sudden changes, giving rise to increasingly complex states with increasing flow speed. Later it was shown by Ruelle and Takens [4] that turbulence can ensue after only a finite number of bifurcations, also observed in the experiments by Gollub and Swinney [5]. Today there is a consensus that information on turbulent dynamics can be extracted from the continuity equation and Navier-Stokes equations. Although the appearance of turbulent flow is chaotic it actually has a degree of regularity which makes it possible to study the transition to turbulence by searching for simple nonlinear solutions, called ECS for exact coherent structures, to the Navier-Stokes equations. These solutions correspond to large scale organized patterns appearing as wavy, low velocity streaks, flanked by staggered streamwise vortices; a classical experimental observation are the wavy unstable structures observed in turbulent boundary layers [6]. The linear perturbation of the ECS has usually only a few unstable dimensions and many stable ones, this quality makes it impossible for the flow to settle onto the ECS and the instability causes the flow to bounce back and forth between these nonlinear solutions. This idea has been reinforced by experiments on pipe flow [7] and numerical nonlinear ECS [8, 9], suggesting that transitional flows are organised around unstable periodic states.

It is ideas from dynamical systems, and in particular an approach called the periodic orbit theory ([10], [11]), that motivates the search of the ECS. The essence behind this theory is that chaotic solutions can appear like the periodic ECS and that turbulent statistics can be reproduced by an expansion of the exact solutions. The question is then if the ECS may be the building blocks of a transitional or turbulent flow. We present ECS for the parallel boundary layer flow, where the idea of the self-sustaining process is used [12]. This technique benefits from an artificial forcing function for driving three-dimensional nonlinear structures.

2 DEFINITIONS

An isothermal boundary layer flow over a flat plate with zero inclination is studied. We present the analysis and results of nonlinear exact coherent structures of the Navier-Stokes equations and the continuity equation. The steady laminar state is the well known Blasius flow $\boldsymbol{U}$=$U(x,y)\mathbf{i}$+$V(x,y)\mathbf{j}$. The cartesian coordinates x, y and z refer to the streamwise, vertical and spanwise coordinate, with the corresponding unit vectors **i**, **j** and **k**. The velocity vector is described by the streamwise, wall-normal and spanwise components or $\boldsymbol{u}$=$u\mathbf{i}$+$v\mathbf{j}$+$w\mathbf{k}$, p the pressure, ν kinematic viscosity and time t. The plate is of infinite length in the streamwise x-direction and the spanwise z-direction, and is located at the wall-normal coordinate y=0. All free-stream parameters use the ∞ as a subscript. The governing equations are non-dimensionalised by the unidirectional free-stream speed U_∞, density ρ and the boundary layer length scale δ=$(\nu x/U_\infty)^{1/2}$. The non-dimensionalised Navier-Stokes equations and the continuity equation are:

$$\boldsymbol{u}_t + \nabla p - \frac{\nu}{U_\infty \delta}\nabla^2 \boldsymbol{u} + (\boldsymbol{u}.\nabla)\boldsymbol{u} = \mathbf{0}, \tag{1}$$

$$\nabla.\boldsymbol{u} = 0, \tag{2}$$

where the $U_\infty\delta/\nu$ is the Reynolds number Re and Re_x=Re^2. The truncated computational outer edge is set to y_∞=40.

2.1 *The laminar Blasius flow*

To define the base flow we use the parallel flow assumption (i.e. $V(x,y)$≡0) of the two-dimensional laminar Blasius solution, with no pressure gradient imposed. Solving the Blasius boundary layer equation $f_{\eta\eta\eta} + \frac{1}{2}ff_{\eta\eta}$=0 we get $U(y)$=f_η, where f=$f(\eta)$ and f_η=$df/d\eta$ with the boundary conditions $f(0)$=0, $f_\eta(0)$=0 and $f_\eta(\eta_{max})$=1. The η is the non-dimensional self-similar coordinate and is defined as η=$y[U_\infty/(\nu x)]^{1/2}$ according to the work by Blasius [13]. To still solve the Blasius equation exactly a forcing ($= -Re^{-1}U_{yy}(y)$) needs to be added to the streamwise momentum equation to compensate for the assumed approximation [14, 15].

2.2 *The perturbation equations*

To find three-dimensional exact coherent structures to the Navier-Stokes equations we perform a bifurcation study. For this we add a perturbation $\boldsymbol{u}' = (u', v', w')$ and p' on the uni-directional laminar base flow, leading to the total flow $\boldsymbol{u} = U\mathbf{i} + \epsilon\boldsymbol{u}'$, where ϵ is an amplitude to be determined. Rather than searching for solutions bifurcating from the laminar Blasius flow, different solutions are discovered by adding a body force $\boldsymbol{f}(y,z)$ with an amplitude f_A to the Navier-Stokes equations, to force nonlinear three-dimensional solutions. The $\boldsymbol{f}$ is constructed from the continuity equation with the correct boundary conditions imposed. The governing equations of the perturbation are:

$$\boldsymbol{u}'_t + \nabla p' - \frac{1}{Re}\nabla^2\boldsymbol{u}' + (K_p\frac{df}{d\eta}.\nabla)\boldsymbol{u}' + (\boldsymbol{u}'.\nabla)K_p\frac{df}{d\eta} + \epsilon\boldsymbol{u}'.\nabla\boldsymbol{u}' - f_A\boldsymbol{f} = \mathbf{0}, \tag{3}$$

$$\nabla.\boldsymbol{u}' = 0. \tag{4}$$

The amplitude K_p is explained below. We set the flow to be perturbation-free in the free-stream or $\boldsymbol{u}$=(1,0,0) and no-slip at y=0, i.e. $\boldsymbol{u}(t, x, y = 0, z)$=$\mathbf{0}$. To ensure invariant flow in the free-stream the additional boundary conditions to be imposed are asymptotically decaying disturbances at the outer edge of the computational domain, or $\boldsymbol{u}'$→0 as y→∞. The perturbation equations are separable in x, z and t since U is a function of y only. We assume $\boldsymbol{u}'$ to be a travelling wave expressed as:

$$\boldsymbol{u}' = \sum_{b=-NX}^{NX} \sum_{j=-NZ}^{NZ} \tilde{\boldsymbol{u}}^{(bj)}(y) e^{Ij\beta z} e^{Ib\alpha(x-ct)} \tag{5a}$$

$$= \sum_{b=-NX}^{NX} \sum_{j=-NZ}^{NZ} \sum_{i=0}^{NY} \hat{\boldsymbol{u}}_{bji} T_i(\gamma(y)) e^{Ij\beta z} e^{Ib\alpha(x-ct)}, \tag{5b}$$

with the imaginary unit $I = \sqrt{-1}$, α and β real wave numbers, the T_i is the Chebyshev polynomial and γ a mapping of y onto the domain $-1 \leq \gamma \leq +1$. The $c = c_r + Ic_i$ is a complex eigenvalue which is forced real for the nonlinear study. Viewing the system in a frame of reference $x - c_r t \rightarrow X$ moving with the wave speed c_r we arrive at a time-independent problem or $\partial_t \rightarrow -c_r \partial_X$. The $\tilde{u}^{(0,0)}(y)$ mode seen in equation 5 requires some extra attention since it is the one mode that does not decay exponentially to zero, and is thus finite at y=y_∞. To satisfy the free-stream conditions shown above the Blasius flow is used to compensate for that in order to ensure $\boldsymbol{u} = (1, 0, 0)$. The particular condition to be imposed is:

$$K_p + \epsilon \tilde{u}^{(0,0)}(y = y_\infty) = 1 \tag{6}$$

and can be realised by decomposing the laminar flow as $U(y) = (1+\epsilon K) f_\eta(\eta) = K_p f_\eta(\eta)$ and then impose the free-stream condition. For laminar flow and infinitesimal disturbances we have K_p=1 and for finite amplitude perturbations $K_p \neq 1$ [14, 15, 16, 17]. For engineering applications one can use the skin friction c_f to relate the nonlinear solutions to laminar and turbulent flows. Formulas for the skin friction coefficient are presented below. The theoretical formula of c_f for the laminar flow is the Blasius law 0.664/Re=0.664$Re_x^{-1/2}$ (see e.g. Schlichting [18]); for turbulent flow the theoretical skin friction coefficient is given by Prandtl [19] and von Karman [20]; for experimental data of turbulence the formula of Dhawan [21] is used:

$$c_f = 0.0576 Re_x^{-1/5}, \tag{7a}$$

$$c_f = 0.0277 Re_x^{-1/7}, \tag{7b}$$

$$c_f^{-1/2} = -0.91 + 5.06 \log_{10} \sqrt{Re_x c_f}. \tag{7c}$$

The spanwise spacing z^+=zu_τ/ν between low speed streaks has proved to have a value of 100 [22], and can be used to determine the relevance of the nonlinear solutions discovered. The u_τ is the friction velocity and is defined in terms of the wall stress and the density as $(\tau/\rho)^{1/2}$.

3 NUMERICAL RESULTS

We present three-dimensional nonlinear exact coherent structures (ECS) with the aim to get insight on their relevance to the transition to turbulence. The following symmetry is imposed:

$$\boldsymbol{Z} : (u, v, w, p)(x, y, z, t) = (u, v, -w, p)(x, y, -z, t). \tag{8}$$

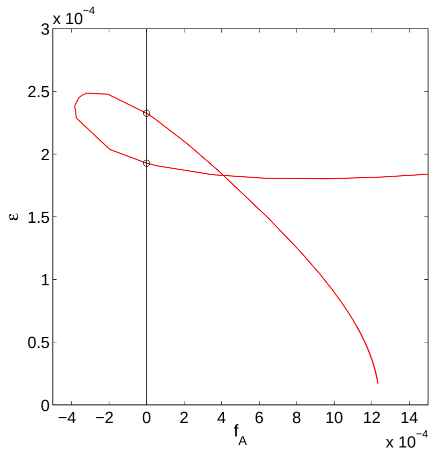

Figure 1: Continuation in f_A from a finite value to the two solutions where it vanishes, pointed out by the black circles. The ϵ is an amplitude of the solution.

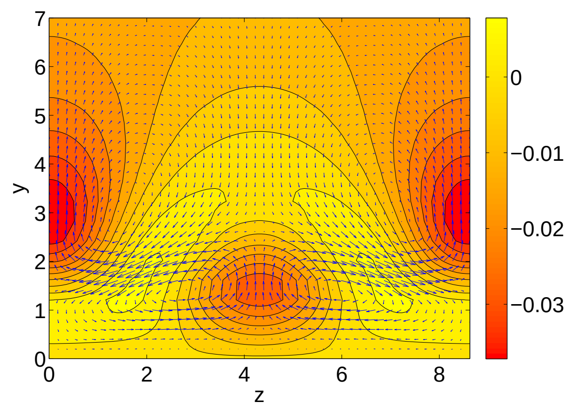

Figure 2: The velocity field (mean over x) of the ECS at (α, β, Re)=$(0.20, 0.728, 400)$. The contour levels represent the streamwise velocity u and range between min(u) and max(u) in steps of 0.0045. The arrows refer to the cross-stream velocity. The color coding goes from most negative (dark) to most positive (light). The (min(u), max(u))=(-0.0372, 0.0089), (min(v), max(v))=(-4.81e-4, 7.86e-4) and (min(w), max(w))=(-0.0011, 0.0011). The figure shows the mean flow in the shape of a four-vortex structure near the wall. The thickness of the Blasius boundary layer flow corresponds to y=5.

With symmetry $\boldsymbol{Z}$ imposed and no symmetry in y gives the following numerical representation:

$$
\begin{bmatrix} u' \\ v' \\ w' \\ p' \end{bmatrix}_{\boldsymbol{Z}} = \sum_{b=0}^{NX}\sum_{j=0}^{NZ}\sum_{i=0}^{NY} \begin{bmatrix} \hat{u}_{bji}T_i(\gamma(y))\cos j\beta z \\ \hat{v}_{bji}T_i(\gamma(y))\cos j\beta z \\ \hat{w}_{bji}T_i(\gamma(y))\sin j\beta z \\ \hat{p}_{bji}T_i(\gamma(y))\cos j\beta z \end{bmatrix} e^{I b\alpha(x-ct)} + c.c \tag{9}
$$

where the c.c is the complex conjugate. The critical Reynolds number for linear instability is around 302 at a spanwise wave number β equal to zero and the streamwise wave number α=0.18. This is the point where Tollmien-Schlichting waves can start to grow. To stay close to that flow situation we fix Re=400 with the objective to bring the artificial forcing f_A (see equation 3) to a vanishing amplitude. This brings us to the unforced Navier-Stokes equations. The bifurcation diagram in figure 1 shows how two solutions can be obtained by bringing f_A to zero. Having reached f_A=0 the parameter space is explored, still keeping Re=400. Moving in β for fixed α=0.20 it is found that the turbulent value z^+=100 is obtained for β=0.728. The mean flow (over x) of that solution is

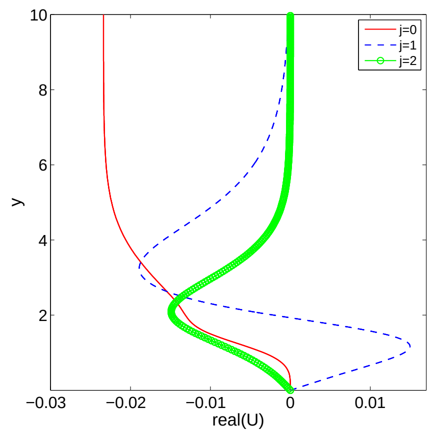

Figure 3: The real eigenfunctions (the imaginary part is identical to zero) of the x-independent part of the perturbation (Fourier mode b=0) at Re=400, α=0.20 and β=0.728. The selected Fourier modes in z are j=0, 1 and 2 (see equation 5), assumed to be the dominant ones. The thickness of the Blasius boundary layer flow corresponds to y=5 and can be used as a reference to realise where in the boundary layer the viscous non-zero part of the eigenfunctions is situated. The $j\neq$0 modes are confined more or less within that layer, while the j=0 mode does not decay with increasing distance from the plate as $y\rightarrow\infty$.

presented in figure 2 showing a four-vortex flow next to the plate. The eigenfunctions of the same solution are shown in figure 3 and 4. Figure 3 shows the x-independent part of the u-component of the perturbation, while figure 4 represents the corresponding x-dependent component. In figure 3 one also gets insight on the additional condition imposed for the K_p shown in equation 6. Here it is seen the finite value of the j=0-mode as $y\rightarrow\infty$. To ensure the correct boundary conditions at the outer edge of the flow domain the K_p is adjusted accordingly. Figure 5 shows the skin friction mapped out in β for α=0.16, 0.20 and 0.33, together with the laminar value and that for turbulent flow of Prandtl (see equation 7a). The turbulent value is included only as a reference, even though it is questionable if it is valid at this low transitional Reynolds number. We see a general trend of an increasing c_f with increasing α. For all 3 α's it is observed that the solutions are situated fairly close

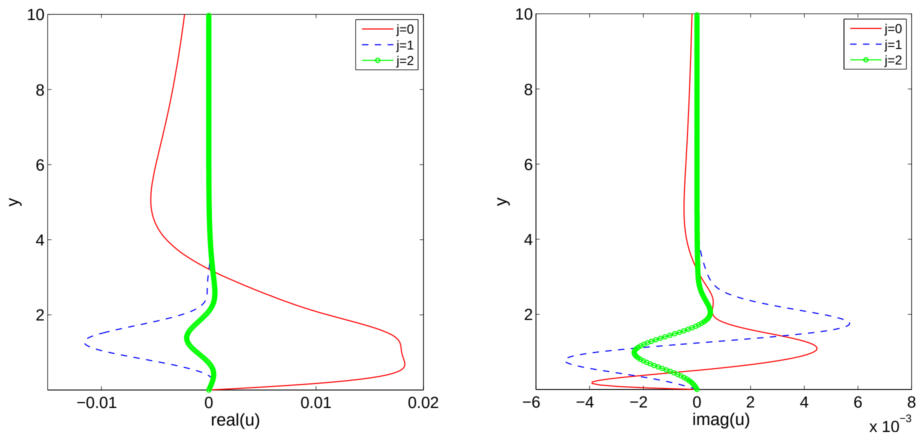

Figure 4: The real and the imaginary eigenfunctions at Re=400, α=0.20 and β=0.728. The selected Fourier modes are b=1 and j=0, 1 and 2 (see equation 5), assumed to be the dominant ones of the x-dependent part of the ECS. The thickness of the Blasius boundary layer flow corresponds to y=5 and can be used as a reference to realise where in the boundary layer the viscous non-zero part of the eigenfunctions is situated. The $j\neq0$ modes are clearly within that layer while the j=0 mode decays slower but still towards zero as $y\to\infty$.

to the laminar state rather than moving towards the turbulent value. Higher values of the Re might change the dependence of the c_f with β. This has not been considered here since we are interested in a transitional flow close to the linear critical point. The solution in figure 2 is especially interesting

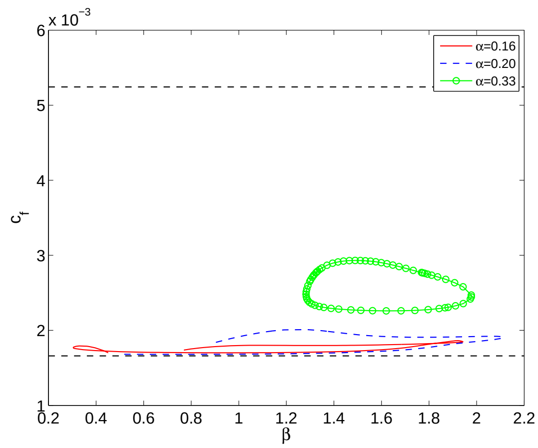

Figure 5: The skin friction c_f as a function of β for Re=400 and 3 values of the streamwise wavenumber α, pointed out in the legends. As a reference the c_f according to Prandtl is added (see equation 7a) and corresponds to the top dashed line, together with the laminar value (the lower horisontal dashed line).

since it corresponds to z^+=100, a value observed in experimental turbulence, but also in studies on the transition (Biau [23]). Keeping z^+ fixed at 100 a whole family of solutions is found, mapped out in the wave number space for Re fixed, for example at α=0.16 and β=0.733.

4 CONCLUSIONS

Nonlinear solutions are presented where some are more interesting than others, having their spanwise spacing z^+ in mind. The solutions discovered are one kind of solutions in a hierarchy of exact solutions to the Navier-Stokes equations, and may be important flow structures to be incorporated in an approach for understanding turbulence called the periodic orbit theory ([10], [11]). In this theory these exact solutions are believed to form the skeleton of chaotic dynamics. Mapping out the solutions in parameter space, e.g. in wave length space, fixing z^+=100 and the Reynolds number to a transitional value gives us the relevant solutions admitted by the Navier-Stokes equations, and may constitute a part of the necessary solutions for giving a macroscopic description of a transitional boundary layer flow over a flat plate.

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